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A MODEL OF ZF WITH AN INFINITE FREE COMPLETE BOOLEAN ALGEBRA†

BY

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ABSTRACT

By a theorem of Gaifman and Hales no model of ZF + AC (Zermelo-Fraenkel set theory plus the axiom of choice) contains an infinite free complete Boolean algebra. We construct a model of ZF in which an infinite free c.B.a. exists.

Notations

ZF is the Zermelo-Fraenkel set theory and ZFA is the theory described in [4, 4.1], which is like ZF except for allowing a set A of atoms (urelements). The language of ZFA contains $=$, $\in$ and the constants 0 (for the empty set) and A (for the set of all atoms). ZF can be identified with ZFA + "$A = 0$". AC is the axiom of choice (not present in ZF). Models of set theories are denoted by $\mathcal{M}, \mathcal{N}, \mathcal{U}, \mathcal{W}, \cdots$ and their underlying sets by $M, N, U, W, \cdots$. V is always the universal class.

$\mathrm{TC}(x)$ is the transitive closure of x; $\mathrm{rank}(x) = \sup\{\mathrm{rank}(y) + 1 \mid y \in x\}$; $\mathcal{P}(y)$ is the power set of y. Following [4, 4.1] we define $\mathcal{P}^\alpha(x)$ by induction on α:

$$\mathcal{P}^0(X) = X$$

$$\mathcal{P}^{\beta+1}(X) = \mathcal{P}^\beta(X) \cup \mathcal{P}(\mathcal{P}^\beta(X)),$$

$$\mathcal{P}^\alpha(X) = \bigcup_{\beta<\alpha} \mathcal{P}^\beta(X) \quad \text{for limit} \quad \alpha,$$

$$\mathcal{P}^\infty(X) = \bigcup_{\alpha} \mathcal{P}^\alpha(X).$$

It is a theorem of ZFA that $V = \mathcal{P}^\infty(A)$. Elements of $\mathcal{P}^\infty(0)$ are called pure sets.

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The cardinality $|X|$ of a set X is assumed to have been somehow defined (in ZFA) so that $|X| = |Y|$ iff there is a 1–1 mapping from X onto Y (cf. [4, 11.2]) *and* that if X is well-orderable then $|X|$ is the least (von-Neumann) ordinal which has a 1–1 mapping onto X. Thus the cardinality $\aleph_0$ of $\omega = \{0, 1, 2, \cdots\}$ is ω.

"B.a." abbreviates "Boolean algebra", "c.B.a."—"complete Boolean algebra". B.a.'s are denoted by $\mathcal{B}, \mathcal{C}$ and their underlying sets by B, C respectively.

Introduction

By a free c.B.a. over a set D we mean a c.B.a. $\mathcal{B}$ together with a mapping $f: D \to B$ such that range (f) generates $\mathcal{B}$ as a c.B.a. and for every c.B.a. $\mathcal{C}$ and mapping $g: D \to C$ there is a complete homomorphism $h: \mathcal{B} \to \mathcal{C}$ such that $g = h \circ f$.

Gaifman and Hales showed that there is no free c.B.a. over D if $|D| \geqq \aleph_0$ (in this form this is a theorem of ZF or even ZFA). The various known proofs of this theorem ([1], [3], [8], [10]) proceed by constructing very large countably generated c.B.a.'s. In all these proofs the set of generators is naturally indexed by pairs (or triples etc.) of natural numbers and to get an ω-sequence of generators a pairing function on ω is needed. This curious common feature of the proofs can be brought into focus by trying to replace ω in them by an arbitrary infinite set Ω. The reader is advised to try this for Solovay's proof in [8] and convince himself (or read in [10, §5]) that the following is established without the axiom of choice.

THEOREM 0.1 (ZF, ZFA). *Let Ω be an infinite set. For each α there is a* c.B.a. *$\mathcal{B}$ generated by a subset of cardinality $|\Omega|^2$ such that $|\alpha| \leqq |B|$. Hence, for every infinite cardinal κ, there is no free* c.B.a. *over a set of cardinality $\geqq \kappa^2$.*

Assuming AC, every infinite set D satisfies $|D| \geqq \aleph_0 (= \aleph_0^2)$ so there is no infinite free c.B.a. (a free c.B.a. over a finite set being finite).

In view of the apparent need for $|\Omega|^2$ in 0.1 the author conjectured ([10, 5.5]) that the non-existence of an infinite free c.B.a. is not provable in ZF. This conjecture is verified in the present paper. We first show (Sec. 2-4) that in the basic Fraenkel model of ZFA ([4, 4.3]) there is a free c.B.A. over the set A of atoms and then (Sec. 5) use the Jech-Sochor first embedding theorem ([4, 6.1]) to get a model of ZF with a free c.B.a. over an infinite set. Transforming this model-construction to a finitary relative consistency proof in any of the usual ways gives

THEOREM. 0.2. *If* ZF *is consistent so is* ZF + "*there exists a free* c.B.a. *over some infinite set.*"

We shall see in Section 5 that some additional statements can be added to ZF in 0.2 without destroying the consistency (e.g.—"there exists a non-principal ultrafilter over ω.") but none of them seems to us particularly interesting in this context. In Section 7 some open problems and further results, as well as a by-product of the work in Section 4, will be discussed.

1. The propositional language

It is useful to have a syntactical necessary and sufficient condition for the existence of a free c.B.a. over any given set D. In the terminology of Gaifman [1] the condition is simply that there is a set of Boolean terms over D which contains a representative for each equivalence class (this is equivalent to saying that the Boolean polynomials over D form a set). It will be necessary for us to formulate the condition a bit more explicitly.

Given a set D, consider sentences built up from the propositional parameters p_i $(i \in D)$ by $-, \wedge, \vee$ (where $\wedge$ and $\vee$ apply to sets of sentences of unlimited cardinality). Following [9, §1] these sentences will also be called Boolean terms (B.t.'s) over D. The depth $d(\phi)$ of a B.t. ϕ is defined by induction as follows:

$$d(p_i) = 0;$$

$$d(-\psi) = d(\psi) + 1;$$

$$d(\wedge X) = d(\vee X) = \sup\{d(\psi) + 1 \mid \psi \in X\}.$$

(We preserve the word "rank" for the set-theoretical notion of rank.)

$\vdash \phi$, where ϕ is a B.t., means that ϕ is provable in the basic formal system of [5, 5.11] equivalently (as is clear from [5, 6.1–6.3]) that $\|\phi\| = 1$ in every Boolean-valued model ("valuation" in the terminology of [9, §1]) in which ϕ is defined. $\phi \equiv \psi$ means $\vdash \phi \leftrightarrow \psi$. Thus $\phi \equiv \psi$ iff $\|\phi\| = \|\psi\|$ in every valuation in which ϕ and ψ are both defined.

THEOREM 1.1. *A necessary and sufficient condition for the existence of a free* c.B.a. *over D is that there is an ordinal α such that every* B.t. *over D is equivalent* ($\equiv$) *to one of depth less than α.*

The proof should be clear to any reader familiar with [1] or with equivalent constructions of free algebras from terms.

We shall also have occasion to use two-valued models. A two-valued model for the propositional language over D is a function $f: D \to \{0, 1\}$. The atomic sentence p_i $(i \in D)$ is considered true in f if $f(i) = 1$, false if $f(i) = 0$. Truth or falsehood of any sentence ϕ in f are defined by induction on ϕ as usual. We let $2^D = \{f \mid f: D \to \{0, 1\}\}$ (the space of two-valued models). For a B.t. (sentence) ϕ over D we let $\mathrm{Mod}(\phi) = \{f \in 2^D \mid \phi$ is true in $f\}$ (the set of models of ϕ). Clearly if $\vdash \phi$ then $\mathrm{Mod}(\phi) = 2^D$ and if $\phi \equiv \psi$ then $\mathrm{Mod}(\phi) = \mathrm{Mod}(\psi)$.

THEOREM 1.2. *If ϕ is countably long (i.e.—the set of subsentences of ϕ is countable) and* $\mathrm{Mod}(\phi) = 2^D$ *then $\vdash \phi$. If ϕ and ψ are both countably long and* $\mathrm{Mod}(\phi) = \mathrm{Mod}(\psi)$ *then $\phi \equiv \psi$.*

The first part is the "countable completeness theorem" ([5, 5.3.2]), which says that every logically valid countably long sentence is provable. The second part of 1.2 is obtained by applying the first part to the sentence $\phi \leftrightarrow \psi$.

The definitions and theorems just outlined can all be formalized in ZFA using a natural identification of sentences with sets (e.g. $p_i = (0, i)$, $-\phi = (1, \phi)$, $\wedge X = (2, X)$, $\vee X = (3, X)$). Then the notions "Boolean term (over a given set)", "valuation (over a given set)," "the value of a B.t. in a valuation," "the depth of a B.t." are Δ_1^{ZFA} (cf. [7]) hence absolute between transitive $\in$-models of ZFA. This is clear from standard closure properties of Δ_1 predicates and operations but one must be careful here to define valuations (= Boolean valued models) without requiring the B.a. of "truth values" to be complete. This is done in [9, §1].

It follows that the statement "$\vdash \phi$" (hence also—"$\phi \equiv \psi$") is Δ_1^{ZFA}. Indeed $\vdash \phi$ iff there *exists* a derivation of ϕ iff $\|\phi\| = 1$ in *all* valuations in which ϕ is defined. (Cf. Gregory [2, §3] for the same argument.) Thus "$\vdash$" and "$\equiv$" are absolute between transitive $\in$-models of ZFA. This absoluteness will play a crucial role in the main argument (Section 3).

2. The model

Our working set theory from now on is $-$ ZFA + AC + "the set A of atoms is infinite countable." Every permutation π of A induces as automorphism π of the universe $\mathcal{V} = \langle V, \in, 0, A\rangle$ $(V = \mathcal{P}^{\infty}(A))$ as explained in [4, 4.2]. For each $E \subseteq A$ let $\mathrm{fix}(E)$ be the set of all permutations π of A satisfying $\pi a = a$ for each $a \in E$. E is said to be a support for x $(x \in V)$ when $\pi x = x$ for every $\pi \in \mathrm{fix}(E)$. x is said to be symmetric when it has a finite support. Let HS be the class of hereditarily symmetric elements, that is $HS = \{x \mid$ every element of

TC$\{x\}$ is symmetric}. The basic Fraenkel model ([4, 4.3]) is just $\mathcal{S} = \langle HS, \in, 0, A\rangle$ and it satisfies each axiom of ZFA.

We shall see that in $\mathcal{S}$ there is a free c.B.a. over the infinite set A, but the proof will not be direct. Rather, it will go through a countable model of ZFA.

By a standard model (for the language of ZFA) we mean in this section a model of the form $\mathcal{X} = \langle X, \in, 0, A\rangle$ where X is a transitive class, $0 \in X$, $A \in X$ (hence $A \subseteq X$). Thus all standard models have the same set of atoms, $\mathcal{V}$ and $\mathcal{S}$ are standard models.

Let $\mathcal{U}$ be a countable standard model of ZFA + AC which is elementarily equivalent to our universe $\mathcal{U}$. Strictly speaking, we assume only that $\mathcal{U}$ (satisfies finitely many axioms of ZFA + AC and) is elementarily equivalent to $\mathcal{V}$ for finitely many formulas, namely—those needed for our considerations. The existence of $\mathcal{U}$ is then well-known by the reflection principle. We denote by $\mathcal{W}$ the basic Frankel model associated with $\mathcal{U}$, that is—$W = \{x \mid x \in U$ and $\mathcal{U} \models$ "x is hereditarily symmetric"$\}$. Clearly $\mathcal{W}$ is a standard model of ZFA. Since $\mathcal{W}$ is defined within $\mathcal{U}$ in the same way that $\mathcal{S}$ is defined within $\mathcal{V}$, $\mathcal{W}$ is elementarily equivalent to $\mathcal{S}$.

It is easy to see that if E is a finite subset of A, $x \in V$ and E supports x then $\mathcal{U} \models$ "E supports x". It follows without difficulty that if $x \in U$ and $x \in HS$ then $x \in W$. The following lemma shows that $W = U \cap HS$ and it will play a key role in Section 3.

LEMMA 2.1. *Every element of W is hereditarily symmetric.*

We postpone proof of this to Section 4.

MAIN THEOREM 2.2. *In $\mathcal{W}$, there is a free* c.B.a. *over A.*

The proof is given in Section 3. Since $\mathcal{W}$ is elementarily equivalent to $\mathcal{S}$, the same statement holds in $\mathcal{S}$, but our proof will use the countability of $\mathcal{W}$.

3. Proof of the main theorem

THEOREM 3.1. *In $\mathcal{W}$, every* B.t. *over A is equivalent to one of depth* $\leqq 4$.

This implies the main theorem (2.2) by 1.1. This section is devoted to the proof of 3.1. All statements, except when modified by "in $\mathcal{W}$" or a similar phrase, are about the real universe $\mathcal{V}$. Recall from Section 1 that the notions "B.t.", "$\vdash$", "$\equiv$" and "depth" are absolute between $\mathcal{U}$ and $\mathcal{W}$.

DEFINITION 3.2. Let $E \subseteq A$; $f, g \in 2^A$. $f \sim_E g$ when there is a permutation $\pi \in \mathrm{fix}(E)$ such that $g = \pi f$.

LEMMA 3.3. *For each $E \subseteq A$, $\sim_E$ is an equivalence relation on 2^A. If ϕ is a sentence* (B.t.) *over A supported by E then* Mod ϕ *is a union of equivalence classes of* $\sim_E$.

PROOF. The first part is obvious. For the second we have to show that if $f \sim_E g$ and $f \in \mathrm{Mod}(\phi)$ then $g \in \mathrm{Mod}(\phi)$. So let ϕ be supported by E and f, $g \in 2^A$, $\pi \in \mathrm{fix}(E)$, $g = \pi f$. Then $\pi\phi = \phi$, hence

$$f \in \mathrm{Mod}(\phi) \Leftrightarrow \pi f \in \mathrm{Mod}(\pi\phi) \Leftrightarrow g \in \mathrm{Mod}\,\phi). \qquad \blacksquare$$

Our plan is as follows: Consider a finite set $E \subseteq A$. Each equivalence class of $\sim_E$ will be represented as $\mathrm{Mod}(\chi)$ for some explicitly written sentence χ of depth $\leqq 3$. For example, if $E = \{a_1, a_2\}$, $a_1 \neq a_2$, a typical χ might "say" that p_{a_1} is true, p_{a_2} is false and p_a is true for infinitely many $a \in A - E$ and false for exactly 7. It will follow from 3.3 that every sentence ϕ supported by E is logically equivalent to a disjunction of such sentences χ. We shall verify that if $\phi \in W$ this disjunction belongs to W and is Boolean-equivalent ($\equiv$) to ϕ, proving 3.1.

We now proceed with the details. Let E be a fixed finite subset of A, $\sim_E$ has already been defined. The set Typ_E of types over E is defined thus:

$$\mathrm{Typ}_E = \{(e, \mu, \nu) \mid e\colon E \to \{0,1\},\ \mu \leqq \omega,\ \nu \leqq \omega$$
$$\text{and at least one of } \mu, \nu = \omega\}.$$

Let $f \in 2^A$. The type of f over E, denoted $\mathrm{typ}_E(f)$, is defined as the triple (e, μ, ν) where $e = f \mid E$ (the restriction of f to E),

$$\mu = |\{a \in A - E \mid f(a) = 1\}|,\quad \nu = |\{a \in A - E \mid f(a) = 0\}|.$$

($|\cdot|$ is the cardinal of the set; for subsets of A it is a natural number on ω since A is countable.) It is clear that $\mathrm{Typ}_E = \{\mathrm{typ}_E(f) \mid f \in 2^A\}$ and for f, $g \in 2^A$, $f \sim_E g \Leftrightarrow \mathrm{typ}_E(f) = \mathrm{typ}_E(g)$. Thus the types over E represent the equivalence classes of $\sim_E$. Let

$$M_\tau = \{f \in 2^A \mid \mathrm{typ}_E(f) = \tau\} \qquad (\tau \in \mathrm{Typ}_E).$$

Next we associate with each $\tau \in \mathrm{Typ}_E$ a sentence (B.t.) $\chi = \chi(E, \tau)$ over A such that $\mathrm{Mod}(\chi) = M_\tau$. χ is defined by cases as follows:

Case 1. $\tau = (e, n, \omega)$ where $e\colon E \to \{0,1\}$, $n < \omega$. Let "F" range over the n-element subsets of $A - E$.

$$\chi = \bigvee_F \bigwedge(\{p_a \mid a \in E, e(a) = 1 \text{ or } a \in F\}$$

$$\cup\{-p_a \mid a \in E, e(a) = 0 \quad \text{or} \quad a \in A - E - F\}).$$

Case 2. $\tau = (e, \omega, n)$. Again let F range over the n-element subsets of $A - E$

$$\chi = \bigvee_F \bigwedge(\{p_a \mid a \in E, e(a) = 1 \quad \text{or} \quad a \in A - E - F\}$$

$$\cup\{-p_a \mid a \in E,\ e(a) = 0 \quad \text{or} \quad a \in F\}).$$

Case 3. $\tau = (e, \omega, \omega)$. The simplest choice for $\chi = \chi(E, (e, \omega, \omega))$ is

$$[\bigwedge(\{p_a \mid a \in E,\ e(a) = 1\} \cup \{-p_a \mid a \in E, e(a) = 0\})]$$

$$\wedge - \bigvee_{n<\omega} (\chi(E, (e, n, \omega)) \vee \chi(E, (e, \omega, n))),$$

where $\chi(E, (e, n, \omega))$ and $\chi(E, (e, \omega, n))$ have been defined in cases 1–2. By writing it explicitly and applying de-Morgan's and other obvious rules one can bring this sentence to an equivalent form of depth $\leqq 3$ (in cases 1–2 inspection shows that $d(\chi) \leqq 3$).

It is clear from the definitions that $\mathrm{Mod}(\chi(E, \tau)) = M_\tau$ for $\tau \in \mathrm{Typ}_E$. Let $X_E = \{\chi(E, \tau)\ \tau \in \mathrm{Typ}_E\}$. Since Typ_E and X_E have been obtained from A and E by absolute (Δ_1^{ZFA}) operations, it is clear that they are in W for each finite $E \subseteq A$.

Now let ϕ be any B.t. over A supported by E. By 3.3 $\mathrm{Mod}(\phi)$ is a union of classes $M_\tau (\tau \in \mathrm{Typ}_E)$, hence there is a set $T \subseteq \mathrm{Typ}_E$ such that $\mathrm{Mod}(\phi) = \mathrm{Mod}(\bigvee_{\tau \in T} \chi(E, \tau))$. In fact we can take

$$T = \{\tau \in \mathrm{Typ}_E \mid \mathrm{Mod}(\chi(E, \tau)) \subseteq \mathrm{Mod}(\phi)\}.$$

In case ϕ is countably long, we have by 1.2,

$$T = \{\tau \in \mathrm{Typ}_E \mid \vdash [\chi(E, \tau) \to \phi]\}$$

and for this T $\phi \equiv \bigvee_{\tau \in T} \chi(E, \tau)$. (Note that the $\chi(E, \tau)$'s are countably long since A is countable).

Now let ϕ be a B.t. over A, $\phi \in W$. Then ϕ is supported by E, for some finite $E \subseteq A$. By the absoluteness of "$\vdash$" the set

$$T = \{\tau \in \mathrm{Typ}_E \mid \vdash [\chi(E, \tau) \to \phi]\}$$

is in W ($T \subseteq X_E$), hence the sentence $\psi = \bigvee_{\tau \in T} \chi(E, \tau)$ is in W. By the above $\phi \equiv \psi$ (in the real universe, hence in $\mathscr{W}$) and ψ is a B.t. over A of depth $\leqq 4$ (since $d(\chi(E, \tau)) \leqq 3$ for $\tau \in \mathrm{Typ}_E$). This completes the proof of 3.1.

4. Absoluteness of hereditary symmetry

We wish to prove Lemma 2.1, which states that if x is hereditarily symmetric in the sense of $\mathscr{U}$ then x is hereditarily symmetric. The proof will give some more information, which may be of independent interest. In fact we shall show that the notion of hereditary symmetry is unchanged when only permutations moving just finitely many atoms are considered. This will immediately imply the absoluteness of the notion.

Theorem 4.2 below has been known to people working on Fraenkel-Mostowski models but to the author's knowledge has never been announced.

We continue to use the notations of Section 2.

DEFINITIONS 4.1. For $E \subseteq A$ let $\mathrm{fix}_\omega(E) = \{\pi \mid \pi \in \mathrm{fix}(E)$ and $\{a \in A \mid \pi(a) \neq a\}$ is finite$\}$. E is a weak support for x ($x \in V$) when $\pi x = x$ for all $\pi \in \mathrm{fix}_\omega(E)$. x is weakly symmetric when x has a finite weak support. x is hereditarily weakly symmetric ($x \in HWS$) when every element of $\mathrm{TC}\{x\}$ is weakly symmetric.

Note that all these notions are absolute between standard models of ZFA (syntactically they are Δ_1^{ZFA}). Thus to get the absoluteness of "hereditarily symmetric" it will suffice to prove in ZFA the following:

THEOREM 4.2. *x is hereditarily symmetric iff x is hereditarily weakly symmetric* ($HS = HWS$).

It is obvious that every support for x is a weak support for x, hence that every symmetric element is weakly symmetric and that $HS \subseteq HWS$. The rest of this section is devoted to the proof in ZFA that $HWS \subseteq HS$.

The idea is to assign to each $x \in HWS$ a non-empty set N_x of "names". Each name can be regarded as a term describing the $\in$-structure of $\mathrm{TC}\{x\}$ in such a way that only the atoms from some finite weak support for x are mentioned (other atoms need not be mentioned because they can be changed without changing x). It is formally simpler to let N_x consist not of terms in the ordinary syntactical sense but just pairs (p, q) consisting of a pure set p ($p \in \mathscr{P}^\infty(0)$) and a sequence without repetitions q of atoms such that range (q) is a weak support for x. There will be an absolute (Δ_1^{ZFA}) defined operation $\mathbf{D}$ that recovers x from any of its names ($x = \mathbf{D}(t)$ for $t \in N_x$). Thus for each $x \in HWS$ there will be p, q as above such that $x = \mathbf{D}(p, q)$. It will follow quite easily that each $x \in HWS$ is symmetric, hence that $HWS \subseteq HS$.

Having outlined the proof we turn to the details. The letters q, q' will always denote finite sequences of distinct atoms, $lh(q)$ is the length of q so that

$$q : \{i \mid 1 \leqq i \leqq lh(q)\} \xrightarrow{1-1} A \, (lh(q) \text{ may be } 0).$$

$\rho(q, q')$ is the sequence r of length $lh(q')$ defined as follows. For $1 \leqq j \leqq lh(q')$, $r(j)$ is the unique i ($1 \leqq i \leqq lh(q)$) such that $q'(j) = q(i)$, if such an i exists, $r(j) = 0$ otherwise. Thus $r = \rho(q, q')$ is a pure set (sequence of natural numbers) and q' is determined by q and r up to permutations in fix (range(q)). For each x let $Q_x = \{q \mid \text{range}(q) \text{ is a weak support for } x\}$. In particular, Q_0 is the set of all finite sequences of distinct atoms.

Define N_x by recursion on x as follows (where q, q' range over Q_0):

Case 1. x is an atom. $N_x = \{((1, i), q) \ 1 \leqq i \leqq lh(q),\ q(i) = x\}$.

Case 2. x is a set and N_y has been defined for $y \in x$. For each $q \in Q_x$ let

$$P_{x,q} = \{(p', \rho(q, q')) \mid (p', q') \in N_y \text{ for some } y \in x\}.$$

Let $N_x = \{((2, P_{x,q}), q) \mid q \in Q_x\}$.

Thus in each case N_x consists of pairs (p, q) with $q \in Q_x$. An easy induction on x shows that whenever $(p, q) \in N_x$, p is a pure set. The idea in case 2 is that, having chosen $q \in Q_x$, we collect all names (p', q') for elements of x but retain from each name only the information which is invariant under permutations in fix (range (q)). [Since $q \in Q_x$, x itself is invariant under these permutations.]

Inspection of the definition shows that N_x always contains exactly one pair (p, q) for each $q \in Q_x$. In particular, $N_x \neq \emptyset$ for weakly symmetric x.

Next we define an operation $\boldsymbol{D}$ (the "denotation" operation on names) such that whenever $x \in HWS$ and $t \in N_x - X = \boldsymbol{D}(t)$. $\boldsymbol{D}(t)$ is defined (for all t) by $\in$-recursion as follows.

Case 1. t has the form $((1, i), q)$, $1 \leqq i \leqq lh(q)$. Then $\boldsymbol{D}(t) = q(i)$.

Case 2. t has the form $((2, P), q)$. Let

$$\boldsymbol{D}(t) = \{\boldsymbol{D}(p', q') \mid (p', \rho(q, q')) \in P\}.$$

Case 3. None of the above. Then $\boldsymbol{D}(t) = 0$.

Clearly $\boldsymbol{D}$ is a well defined operation in ZFA (in fact it is Δ_1^{ZFA}). Remember that ZFA contains the primitive constant "A", hence "Q_0" can be regarded as a defined constant; q and q' range over Q_0 in the definition of $\boldsymbol{D}$.

LEMMA 1. *If $x \in HWS$ then $(\forall t \in N_x)\boldsymbol{D}(t) = x$.*

PROOF. Induction on x. The case of an atom is clear by inspection of the definitions. So let $x \in HWS$ be a set (so $x \subseteq HWS$) and assume that for all $y \in x$, $(\forall t' \in N_y)\boldsymbol{D}(t') = y$. Let $t \in N_x$. For some $q \in Q_x$, t has the form $((2, P_{x,q}), q)$ where $P_{x,q}$ is defined as above. By definition then

$$\boldsymbol{D}(t) = \{\boldsymbol{D}(p', q') \mid (p', \rho(q, q')) \in P_{x,q}\}.$$

To show that $\boldsymbol{D}(t) = x$ we have to show:

(i) Each $y \in x$ is of the form $\boldsymbol{D}(p', q')$ for some p', q' such that $(p', \rho(q, q')) \in P_{x,q}$;
(ii) whenever $(p', \rho(q, q')) \in P_{x,q}$, $\boldsymbol{D}(p', q') \in x$.

Proof of (i): Let $y \in HWS$ so choose some $q' \in Q_y$ and let p' be the unique set such that $(p', q') \in N_y$. Then $(p', \rho(q, q')) \in P_{x,q}$ be definition of $P_{x,q}$ and $\boldsymbol{D}(p', q') = y$ by induction hypothesis.

(ii): Here a symmetry argument is needed. Assume that p', q' are such that $(p', \rho(q, q')) \in P_{x,q}$. Then there exist p'', q'' such that $(p'', q'') \in N_y$ for some $y \in x$ and $(p', \rho(q, q')) = (p'', \rho(q, q''))$. Thus $p' = p''$ and $\rho(q, q') = \rho(q, q'')$. It follows from the definition of $\rho(q, \cdot)$ that there exists some $\pi \in \mathrm{fix}_\omega$ (range (q)) such that $q' = \pi q''$. But $(p', q'') = (p'', q'') \in N_y$, hence (applying π) $(p', q') \in N_{\pi(y)}$ (since $y \to N_y$ is a defined operation of ZFA). But $q \in Q_x$ so $\pi x = x$, hence $\pi y \in x$ (as $y \in x$). By induction hypothesis, $\boldsymbol{D}(p', q') = \pi(y)$, hence $\boldsymbol{D}(p', q') \in x$, proving (ii).

We have shown that $\boldsymbol{D}(t) = x$ for any $t \in N_x$. This completes the induction step and the proof. ■

LEMMA 2. *If $x \in HWS$ and $E \subseteq A$ is finite and weakly supports x then E supports x.*

PROOF. Let q enumerate E without repetitions. Then range (q) weakly supports x so $q \in Q_x$. Let p be a pure set such that $(p, q) \in N_x$ (we have seen that p exists, and is unique, for $q \in Q_x$). By Lemma 1, $x = \boldsymbol{D}(p, q)$. Thus if $\pi \in \mathrm{fix}(E)$ then $\pi q = q$, hence

$$\pi x = \pi \boldsymbol{D}(p, q) = \boldsymbol{D}(\pi p, \pi q) = \boldsymbol{D}(p, q) = x.$$

This shows that E is a (strong) support for x. ■

LEMMA 3. $HWS \subseteq HS$.

PROOF. By Lemma 2 each $y \in HWS$ is symmetric. Thus

$$x \in HWS \Rightarrow (\forall y \in \mathrm{TC}\{x\})(y \in HWS) \Rightarrow (\forall y \in \mathrm{TC}\{x\})$$
$$(y \text{ is symmetric}) \Rightarrow x \in HS. \quad ■$$

The proof of 4.2 in ZFA, without AC, is thus complete and Lemma 2.1 follows immediately from the absoluteness of the notion "hereditarily weakly symmetric" between standard models of ZFA.

The proof of the main theorem (2.2), is now complete too.

Note on terminology: The phrase "weakly symmetric" is ad hoc. In a systematic terminology we should have replaced "symmetric" and "weakly symmetric" by "symmetric with respect to the group $\mathcal{G}$ (and the filter of subgroups $\mathcal{F}$)" and "symmetric with respect to $\mathcal{G}'$ (and $\mathcal{F}'$)" respectively, where $\mathcal{G}$ is the group of all permutations of A, $\mathcal{G}'$ the subgroup of permutations moving only finitely many elements (and $\mathcal{F}, \mathcal{F}'$ are the obvious filters). See [4, 4.2].

5. Transition to a model of ZF

We wish to go from the ZFA-model $\mathcal{W}$ to a standard model of ZF with an infinite free c.B.a. Since $\mathcal{W}$ is a permutation (Frenkel-Mostowski) submodel of $\mathcal{U}$, which is a standard countable model of ZFA + AC, the Jeck-Sochor first embedding theorem ([4, 6.1]) is applicable. Denoting by $\mathcal{M}$ the kernel of $\mathcal{U}$ ($\mathcal{M} = \langle M, \in \rangle$ where $M = U \cap \mathcal{P}^{\infty}(0)$), the theorem asserts the existence of a symmetric (Cohen) extension $\mathcal{N} = \langle N, \in \rangle$ of $\mathcal{M}$ and a set $\tilde{A} \in N$ such that $\mathcal{P}^{\omega}(A)^{(\mathcal{W})}$ is $\in$-isomorphic to $\mathcal{P}^{\omega}(\tilde{A})^{(\mathcal{N})}$. $\mathcal{N}$ is a standard model of ZF and it remains only to show that in $\mathcal{N}$ there is a free c.B.a. over $\tilde{A}$ (which is clearly infinite).

For brevity let $T = \mathcal{P}^{\omega}(A)^{(\mathcal{W})}$, $\tilde{T} = \mathcal{P}^{\omega}(A)^{(\mathcal{N})}$. As seen from the proof in [4, 6.1] an isomorphism $\sim$ of $\langle T, \in \rangle$ and $\langle \tilde{T}, \in \rangle$ is defined so that the image of A is $\tilde{A}$ (i.e.—$\tilde{A} = \{\tilde{a} \mid a \in A\}$) and for every set $x \in T$, $\tilde{x} = \{\tilde{y} \mid y \in x\}$.

Note that if ϕ is a B.t. of finite depth over $A(\tilde{A})$ and $\phi \in W$ ($\phi \in N$) then $\phi \in T$ ($\phi \in \tilde{T}$ respectively). Here we use the definitions $p_i = (0, i)$, $-\phi = (1, \phi)$, $\wedge X = (2, X)$, $\vee X = (3, X)$. An easy induction on n, using the above properties of $\sim$, shows that for every $x \in T$: x is a B.t. of depth $\leqq n$ over A iff $\tilde{x}$ is a B.t. of depth $\leqq n$ over $\tilde{A}$.

LEMMA. *If $\phi, \psi \in W$ are B.t.'s of finite depth over A then $\phi \equiv \psi$ iff $\tilde{\phi} \equiv \tilde{\psi}$.*

PROOF. With each valuation over A (i.e. — Boolean-valued model for the propositional language over $\{p_a (a \in A\}$) we associate a valuation over $\tilde{A}$ by letting the new value of $p_{\tilde{a}}$ be the old value of p_a (for all $a \in A$). An easy induction on depth shows that for any B.t. $\phi \in W$ of finite depth over A, the new value of $\tilde{\phi}$ is the old value of ϕ (or both are undefined, which may happen if the B.a. is incomplete—see [9, §1]). Now, if $\phi \not\equiv \psi$ then there is a valuation over A in which ϕ, ψ are defined and have different values. In the induced valuation over $\tilde{A}$, $\tilde{\phi}$ and $\tilde{\psi}$ are defined and have different values. This shows that $\phi \not\equiv \psi \Rightarrow \tilde{\phi} \not\equiv \tilde{\psi}$ and the proof of ($\tilde{\phi} \not\equiv \tilde{\phi} \Rightarrow \phi \not\equiv \psi$) is similar. ∎

Combining the lemma with the facts stated before it we can argue as follows: Let $\chi \in N$ be a B.t. of depth 5 over $\tilde{A}$. $\chi = \tilde{\phi}$ for some B.t. $\phi \in W$ of depth 5 over A. By 3.1 there is in W some B.t. ψ of depth $\leqq 4$ over A such that $\phi \equiv \psi$. $\tilde{\psi}$ is then a B.t. of depth $\leqq 4$ over $\tilde{A}$, $\tilde{\psi} \in N$ and $\chi = \tilde{\phi} \equiv \tilde{\psi}$. Thus in $\mathcal{N}$ the following statement holds (by the absoluteness of "$\equiv$"):

Every B.t. over $\tilde{A}$ of depth 5 is equivalent to some B.t. over $\tilde{A}$ of depth $\leqq 4$.

Thus in $\mathcal{N}$ the B.t's of depth $\leqq 4$ over $\tilde{A}$ are closed, up to equivalence, under the operations —, $\wedge, \vee$ and an easy induction shows that (in $\mathcal{N}$) every B.t. over $\tilde{A}$ is equivalent to one of depth $\leqq 4$. By 1.1 there is in $\mathcal{N}$ a free c.B.a. over $\tilde{A}$.

The reader will easily check that in $\mathcal{N}$ the set $\tilde{A}$ has some other peculiar properties: It has only finite and cofinite subsets, it cannot be linearly ordered etc. (cf. [4, 4.6, Problems 3,7 and 6.3, Problems 3,5]). Also, by using the embedding theorem with $\omega + \omega$ (choosing $\mathcal{N}_1$ and A_1, $A_1 \in \mathcal{N}_1$, so that $\mathcal{P}^{\omega+\omega}(A)^{(\mathcal{W})}$ is isomorphic to $\mathcal{P}^{\omega+\omega}(A_1)^{(\mathcal{N}_1)}$), we can transfer more properties. For example, since the kernel $\mathcal{M}$ of $\mathcal{U}$ and $\mathcal{W}$ is a model of ZFC there is in $\mathcal{M}$ a non-principal ultrafilter u over ω. Since $\operatorname{rank}(u) = \omega + 1$ and since $\mathcal{M}$ and $\mathcal{N}_1$ have the same sets of rank less than $\omega + \omega$ (as is easy to check), u is a non-principal ultrafilter over ω in $\mathcal{N}_1$. Thus various stronger (but apparently not very interesting) versions of 0.2 can be proved (see also Section 7).

6. Alternative approaches

I. Our proof that there is a free c.B.a. over A in $\mathcal{W}$, hence in $\mathcal{S}$, is indirect and uses essentially the countability of (elements of) W and the fact (2.1) that for every B.t. $\phi \in W$ there is a finite set $E \subseteq A$ such that $\pi\phi = \phi$ for every $\pi \in \operatorname{fix}(E)$ (this is needed to show if $f \sim_E g$ then $f \in \operatorname{Mod}(\phi) \Leftrightarrow g \in \operatorname{Mod}(\phi)$). It would be interesting to see a more direct proof which will give the existence a free c.B.a. over A in $\mathcal{S}$ without going through countable models. A possible approach is to give a more explicit description of the form of hereditarily symmetric B.t.'s over A, which will show that each such B.t. of depth 5 is equivalent to one of depth $\leqq 4$.

II. As long as one is looking for a ZFA model only the work of Section 4 could be dispensed with as follows: Let $\mathcal{U}$ be as in Section 2 and let $\mathcal{W}'$ be the least standard model of ZFA (in the sense of Section 2) containing the ordinals of $\mathcal{U}$. That is, $W' = L_{\alpha_0}(A)$ where α_0 is the least ordinal not in U and $L_\alpha(A)$ is defined by induction on α: $L_0(A) = A$, $L_{\beta+1}(A) = \{X \subseteq L_\beta(A) \mid X$ is first order definable in $\langle L_\beta(A), \in, 0, A\rangle$ from parameters in $L_\beta(A)\}$, $L_\alpha(A) = \bigcup_{\beta<\alpha} L_\beta(A)$ for limit α.

Here it is clear that $W' \subseteq HS$ (as $\mathcal{S}$ is a standard model of ZFA containing all ordinals). Thus we could prove the main theorem for $\mathcal{W}'$. without relying on Section 4. The difficulty would move to the stage of getting a ZF-model. To apply the Jech-Sochor embedding theorem to $\mathcal{W}'$ we have to know that there exist a countable standard model $\mathcal{U}'$ of ZFA + AC and (in $\mathcal{U}'$) a group $\mathcal{G}$ of permutations of A and a normal filter $\mathcal{F}$ of subgroup of $\mathcal{G}$, such that $\mathcal{W}'$ consists exactly of the $(\mathcal{G}, \mathcal{F})$ hereditarily-symmetric elements of $\mathcal{U}'$. Such a $\mathcal{U}'$ can be found for $\mathcal{W}'$ just defined but the work involved is not very different from what we did in Section 4.

Thus at present the proof presented in Sections 2–5 is the best we have. Yet the author will not be surprised if a study of the more general questions presented in Section 7 leads to a considerably simpler proof of the result of this paper.

7. Concluding remarks and problems

The argument of Section 4 establishes a little more than was stated. It shows (as a theorem of ZFA) that every element of HWS is obtained from a pure set and a finite sequence of atoms by the Δ_1^{ZFA}-operation $\boldsymbol{D}$. Thus HWS is the least model containing all pure sets and all atoms.

It seems that in many cases where a Fraenkel-Mostowski model $\mathcal{X}$ is defined by a group $\mathcal{G}$ and a filter of subgroups $\mathcal{F}, \mathcal{X}$ is actually the least model containing all pure sets, all atoms and some non-pure sets which were intended to be put in it. For special models this should be provable by arguments similar to that of Section 4. Is there a general theorem of this kind? A similar problem can be posed for symmetric (Cohen) extensions of ZFC models.

The remaining remarks and problems concern the Gaifman-Hales theorem and its extensions. Consider (in ZF or ZFA) the following classes of cardinals:

$C^0 = \{\kappa \mid \kappa$ is an infinite cardinal$\}$,

$C^1 = \{\kappa \mid$ there is no free c.B.a. over a set of cardinality $\kappa\}$,

$C^2 = \{\kappa \mid$ every B.a. has a complete embedding in some c.B.a. generated by $\leqq \kappa$ generators$\}$.

In ZFA we know that $\aleph_0 \in C^1$; Kripke's embedding theorem states that $\aleph_0 \in C^2$ and is provable in ZF (see [6], [10]). In ZFA: $\aleph_0 \in C^2 \Leftrightarrow$ the set A of atoms has a 1–1 mapping into a pure set $\Leftrightarrow$ every set has a 1–1 mapping into a pure set. (The proof is easy.) Thus "$\aleph_0 \in C^2$" is not a theorem of ZFA but is a theorem of ZF and of ZFA + AC (this was noted by S. Kripke). It is obvious (in ZFA) that $C^2 \subseteq C^1 \subseteq C^0$ and AC implies $C^2 = C^1 = C^0$; we have seen (0.2) that $C^1 \subsetneq C^0$ is consistent with ZF. Obvious questions are now to get, in ZF, more information about membership in C^1, C^2 and determine whether "$C^2 \subsetneq C^1 = C^0$", "$C^2 = C^1 \subsetneq C^0$", "$C^2 \subsetneq C^1 \subsetneq C^0$" are all consistent with ZF and how "$C^1 = C^0$" is related to other weak forms of AC. What we know now in ZFA is mainly that $\kappa^2 \in C^1$ for every infinite κ. The author has recently proved in ZFA that $|D| \in C^1$ whenever D is infinite and linearly orderable. In ZF we know also that $\kappa^2 \in C^2$ for every infinite cardinal κ (by [9, 5.7]).

Somewhat more is known (see [11] about whether the set of atoms A satisfies $|A| \in C^1$ in the ordinary Fraenkel-Mostowski models where each element has a finite support. If the permutation group is the group of automorphisms of some given first order structure $\mathscr{A} = \langle A, \cdots \rangle$, then $|A| \in C^1$ in the induced Fraenkel-Mostowski model iff $\mathscr{A}$ satisfies some simple model theoretic condition (in the "real universe", where A is countable). Also an analogue of the result of Section 4 holds. The proofs will appear elsewhere.

One last remark: It is not clear that the mathematical content of the work is best brought out in the form of (relative) consistency or independence results. Perhaps it is more useful to study (in ZFC or ZFA + AC) Boolean terms and algebras which are hereditarily symmetric over a set A in a suitable sense. Does the existence of a "free hereditarily symmetric c.B.a." over an infinite set have any mathematical applications?

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